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LOGARITHMIC DERIVATIVES OF TWO NORMALIZING FUNCTIONS

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ABSTRACT

If the unit vector x in \mathbb{R}^q has a probability density proportional to $\exp \kappa \mu'x$, or $\exp \kappa(\mu'x)^2$, then the statistical theory for x depends largely upon the logarithmic derivatives $A_q(\kappa)$ and $B_q(\kappa)$ of

$$\int_{-1}^1 e^{\kappa t(1-t^2)^v} dt \quad \text{and} \quad \int_{-1}^1 e^{\kappa t^2(1-t^2)^v} dt$$

where $v=(q-3)/2$. This paper gives a self-contained study of the functions $A_q(\kappa)$, $B_q(\kappa)$, of the computational problems of calculating these functions, of solving $y=A_q(\kappa)$ and $y=B_q(\kappa)$, and of finding the variance stabilizing transformations $\int^\kappa A_q(\kappa)^{1/2} d\kappa$, $\int^\kappa B_q(\kappa)^{1/2} d\kappa$ are also discussed.

Key words: Fisher-von Mises distribution, Scheidegger-Watson distribution, power series, asymptotic expansions, variance stabilizing transformations, Bessel functions, Kummer functions, Riccati equation.



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1. INTRODUCTION

Two probability densities on the surface Ω_q of the unit ball in \mathbb{R}^q are given by

$$f_1(x) = a_q^*(\kappa)^{-1} \exp \kappa \mu'x \quad (1.1)$$

$$f_2(x) = b_q^*(\kappa)^{-1} \exp \kappa(\mu'x)^2 \quad (1.2)$$

when $\mu'x$ is the scalar product of the two unit vectors x and μ . μ is called the modal direction and κ a concentration parameter. In (1.1) $\kappa \geq 0$. In (1.2), κ may be any real number. If $\kappa \rightarrow 0$ either density becomes uniform.

The statistical theory of (1.1), the Fisher-von Mises distribution, has been studied most recently by Watson (1981a, 1981b). The statistical theory of (1.2), the Scheidegger-Watson distribution, has been studied for general q in Watson (1981c). It is found that the theory turns upon

$$A_q(\kappa) = a_q^*(\kappa)/a_q^*(\kappa) , \quad (1.3)$$

$$B_q(\kappa) = b_q^*(\kappa)/b_q^*(\kappa) , \quad (1.4)$$

respectively. The integral definitions of a_q^* and b_q^* can be simplified by writing $x = t\mu + (1-t^2)^{1/2}\xi$ where $t = \mu'x$, and ξ is a unit vector orthogonal to μ . Then, writing

$$\omega_q = \text{area of } \Omega_q = 2\pi^{q/2}/\Gamma(q/2) , \quad (1.5)$$

we have

$$a_q^*(\kappa) = \omega_{q-1} a_q(\kappa) , \quad b_q^*(\kappa) = \omega_{q-1} b_q(\kappa)$$

where

$$a_q(\kappa) = \int_{-1}^1 e^{\kappa t(1-t^2)^{(q-3)/2}} dt, \quad (1.6)$$

$$b_q(\kappa) = \int_{-1}^1 e^{\kappa t^2(1-t^2)^{(q-3)/2}} dt, \quad (1.7)$$

and

$$A_q(\kappa) = a'_q(\kappa)/a_q(\kappa), \quad B_q(\kappa) = b'_q(\kappa)/b_q(\kappa) \quad (1.8)$$

The distribution (1.1) originated in Statistical Physics with Langevin's work on magnetism. (Our nomenclature refers only to the first uses of (1.1) for statistical inference.) Dyson, Lieb and Simon (1978) in proving the existence of spontaneous magnetization at sufficiently low temperatures needed some properties of $\log a_q(\kappa)$. Suppose $h(y)$ is defined by

$$h(y) = \log \int_{\Omega_q} \exp y'x \, d\mu(x)$$

where $\mu(x)$ is any measure on Ω_q . Then $h(y)$ is a convex function of y . If c is the largest eigen value of $[\partial^2 h / \partial y_i \partial y_j]$, $\frac{1}{2} c y'y + h(y)$ is concave in y . If μ is the uniform measure on Ω_q , $c = q^{-1}$, a proof of which is given below. Their first result is basic for Laplace transforms -- see e.g. Barndorff-Nielsen (1978).

Schou (1978) gave some properties of and expansions for $A_q(\kappa)$ by recognizing that $a_q(\kappa)$ is proportional $I_{(q/2)-1}(\kappa) \kappa^{-(q/2)-1}$ and using the properties of the modified Bessel function of the first kind, $I_\nu(\kappa)$. Watson (1956) gave some results on $B_q(\kappa)$, $q=3$.

We will derive all the properties of $A_q(\kappa)$ (Section 2) and $B_q(\kappa)$ (Section 3) directly, rather than as the ratios a_q'/a , b_q'/b . More properties and more terms in the expansions will be given than heretofore. Furthermore in practice one needs to be able to compute, any q and κ , $A_q(\kappa)$, $B_q(\kappa)$, solve the equations $y = A_q(\kappa)$, $y = B_q(\kappa)$ and to apply the variance stabilizing transformations (derived in Watson 1981a, 1981b) $g_q(\kappa) = \int^{\kappa} A_q(\kappa)^{1/2} d\kappa$, $h_q(\kappa) = \int^{\kappa} B_q(\kappa)^{1/2} d\kappa$ numerically. These matters are discussed in Section 4.

While we have wished to emphasize that it is A_q and B_q rather than a_q and b_q which matter, we have for completeness included all known properties of the functions a_q and b_q . The former is associated with the modified Bessel function of the first kind $I_\nu(z)$ and the latter to the less well known Kummer function of the first kind $M(a,b,z)$.

2. The functions $a_q(\kappa)$, $A_q(\kappa)$ and its inverse.

Consider the functions of $\kappa \geq 0$.

$$a_q(\kappa) = \int_{-1}^1 e^{\kappa t(1-t^2)^{(q-3)/2}} dt, \quad q \geq 2, \quad (2.1)$$

$$A_q(\kappa) = a_q'(\kappa)/a_q(\kappa). \quad (2.2)$$

For the applications now envisaged q is integer ≥ 2 but the following results are true for real $q \geq 2$.

It is clear that if $q_1 > q_2$, $0 < a_{q_1}(\kappa) < a_{q_2}(\kappa)$. It is simpler to work with

$$y_v(\kappa) = a_q(\kappa), \quad v = (q-3)/2 \geq -\frac{1}{2}, \quad (2.3)$$

$$M_v(\kappa) = y_v'(\kappa)/y_v(\kappa) = A_q(\kappa) \quad (2.4)$$

Examination of

$$y_v'(\kappa) = \int_{-1}^1 e^{\kappa t(1-t^2)^v} dt$$

shows that

$$0 \leq y_v'(\kappa) \leq y_v(\kappa) \quad (2.5)$$

whence

$$0 \leq A_q(\kappa) = M_v(\kappa) \leq 1 \quad (2.6)$$

Elementary manipulations show that

$$y_v' = \frac{\kappa}{2(v+1)} y_{v+1} \quad (v > -1), \quad (2.7)$$

$$\kappa y'_v = -(2v+1)y_v + 2v y_{v-1}, \quad (v > 0), \quad (2.8)$$

$$\kappa y'_v + 2(v+1)y'_v - \kappa y_v = 0, \quad (v \geq -\frac{1}{2}). \quad (2.9)$$

It may be verified that (2.9) has a solution proportional to $I_{v+\frac{1}{2}}(\kappa) \kappa^{-(v+\frac{1}{2})}$ whence

$$a_q(\kappa) = (2\pi)^{q/2} I_{q/2-1}(\kappa) \kappa^{-q/2+1} \quad (2.10)$$

Here $I_\nu(\kappa)$ is the modified Bessel function of the first kind as defined in Watson (1952). Thus as $\kappa \rightarrow 0$,

$$a_q(\kappa) = (2\pi)^{q/2} \kappa^{q/2-1} \sum_{r=0}^{\infty} \frac{(\kappa/2)^{q/2-1+2r}}{r!(q/2-1+r)!} \quad (2.11)$$

while as $\kappa \rightarrow \infty$,

$$a_q(\kappa) = (2\pi)^{q/2-1} \kappa^{(q-3)/2} e^{\kappa} (1 + o(\kappa^{-1})) \quad (2.12)$$

Since

$$A'_q(\kappa) = a''_q/a_q - (a'_q/a_q)^2, \quad (2.13)$$

$$a''_q < a_q,$$

it follows that

$$A'_q(\kappa) + A_q^2(\kappa) < 1. \quad (2.14)$$

Using the Cauchy inequality, $a''_q a_q \geq (a'_q)^2$, in (2.13) and the result of Dyson, Lieb and Simon (1978), we have

$$0 \leq A'_q(\kappa) \leq 1/q \quad (2.15)$$

Returning the probabilistic background, $t = \mu'x$ where x has the distribution (1) so that

$$A_q(\kappa) = Et, \quad (2.16)$$

$$A_q^-(\kappa) = Et^2 - (Et)^2 = \text{var } t, \quad (2.17)$$

so that (2.6) and (2.15) are statistically obvious. For $A_q^-(\kappa) = \text{var } t$ will be a maximum when $\kappa=0$ because of (2.18) below. It is then equal to $1/q$ by a direct easy calculation. Further since

$$\begin{aligned} A_q^{--}(\kappa) &= \frac{a_q^{---}}{a_q} - 3 \frac{a_q^{--}a_q^-}{a_q^2} + 2\left(\frac{a_q^-}{a_q}\right)^3, \\ &= Et^3 - 3 Et^2 Et + 2(Et)^3, \\ &= E(t-Et)^3, \end{aligned}$$

so from the evident skewness of the distribution of t for $\kappa > 0$ it follows that

$$A_q^{--}(\kappa) \leq 0. \quad (2.18)$$

Thus the function $A_q(\kappa)$ is non-decreasing and convex on $(0, \infty)$ taking its minimum at $\kappa=0$ and maximum as $\kappa \rightarrow \infty$ in the range $[0, 1]$. While we will not want $A_q(\kappa)$ for negative κ , it is helpful to observe that $A_q(\kappa)$ is an odd function of κ . It is easy to show that $A_q, A_q^-, A_q^{--}, \dots$ are the successive cumulants of $t = \mu'x$, a fact which is used in Watson (1981d) to derive Edgeworth expansions for certain limiting distributions.

Integrating a_q^{--} by parts and using (2.3), we find that $A_q(\kappa)$ satisfies the Riccati equation

$$A_q^-(\kappa) = 1 - A_q^2(\kappa) - \frac{q-1}{\kappa} A_q(\kappa) , \quad (2.19)$$

as found via Bessel functions by Schou (1978). Similar manipulations show that

$$A_q(\kappa) = \frac{A}{A_{q-2}(\kappa)} - \frac{q-2}{\kappa} , \quad (q \geq 3) , \quad (2.20)$$

as in Schou (1978)(A1). By (2.6), (2.15), and (2.8) it follows that

$$A_q^-(\infty) = 0 , \quad A_q(\infty) = 1 \quad (2.21)$$

Inspection of $A_q(\kappa)$ as $\kappa \rightarrow 0$ shows $A_q(0) = 0$ so

$$A_q^-(\infty) = \frac{1}{q} , \quad A_q(0) = 0 \quad (2.22)$$

We now give more detailed expansions for $A_q(\kappa)$ as $\kappa \rightarrow 0$ and $\kappa \rightarrow \infty$ by solving (2.19). The difference equation will be used later to suggest how to tabulate $A_q(\kappa)$ for various values of q .

For $q=3$, there is a simple explicit formula

$$A_3(\kappa) = \coth \kappa - \kappa^{-1} \quad (2.23)$$

For $q=2$, we may write

$$A_2(\kappa) = I_1(\kappa)/I_0(\kappa) \quad (2.24)$$

since generally

$$A_q(\kappa) = I_{q/2+1}(\kappa)/I_{q/2-1}(\kappa) \quad (2.24')$$

For $q=2$, the zeros of I_0 nearest to the origin are at $\pm 2.41 i$ (approx.). For $q=3$, the zeros of $I_{1/2}$ nearest to the origin are at $\pm 3.81 i$ (approx.). These

comments help to define the radius of convergence of the series expansion (2.25) below.

For $\kappa \rightarrow 0$, we may substitute $a_1\kappa + a_3\kappa^3 + a_5\kappa^5 + \dots$ for the (odd) function $A_q(\kappa)$ in (2.19). We then find that

$$A_q(\kappa) = \frac{1}{q} \kappa - \frac{1}{q^2(q+2)} \kappa^3 + \frac{2}{q^3(q+2)(q+4)} \kappa^5 + O(\kappa^7) \quad (2.25)$$

where the first two terms agree with Schou (1978).

For $\kappa \rightarrow \infty$, we may substitute a series in powers of κ^{-1} in (2.19) and find that

$$A_q(\kappa) = 1 - \frac{q-1}{2} \frac{1}{\kappa} + \frac{(q-1)(q-3)}{8} \frac{1}{\kappa^2} + \frac{(q-1)(q-3)}{8} \frac{1}{\kappa^3} + O\left(\frac{1}{\kappa^4}\right) \quad (2.26)$$

where the first three terms agree with Schou (1978). For $q=3$ the terms in κ^{-2} and κ^{-3} are zero. But from (2.23), we may write

$$A_3(\kappa) = \left(1 - \frac{1}{\kappa} + \frac{\kappa+1}{\kappa} e^{-2\kappa}\right) (1 - e^{-2\kappa})^{-1} \quad (2.27)$$

so we see that $A_3(\kappa) - 1 + \frac{1}{\kappa}$ is exponentially small.

From the form of (1) it is obvious that maximum likelihood leads to equations for κ of the form $y = A(\kappa)$ so that we need $\kappa = A_q^{-1}(y)$ for all y in $[0,1]$. The equation (2.19) can be rewritten

$$\{(1-y^2)\kappa - (q-1)y\} \frac{d\kappa}{dy} = \kappa \quad (2.28)$$

Since the inverse function will also be odd we may put $\kappa = a_1 y + a_3 y^3 + \dots$ in (2.28) for y small. We find then, as $y \rightarrow 0$,

$$\kappa = q y + \frac{q^2}{(q+2)} y^3 + \frac{q^3(q+8)}{(q+2)(q+4)} y^5 + o(y^7) \quad (2.29)$$

Alternatively we could revert the series (2.25) to find (2.29). To find κ when y is near unity we may revert (2.26) using $z = 1-y = 1-A_q(\kappa)$ or set $z = 1-y$ in (2.19) and solve for a series in inverse powers of z . Either way we find that the solution of $y = A_q(\kappa)$ for y near unity is defined by

$$\frac{1}{\kappa} = \frac{2}{q-1} (1-y) + \frac{q-3}{(q-1)} (1-y)^2 + \frac{q-3}{(q-1)^2} (1-y)^3 + o((1-y)^4) \quad (2.30)$$

The case $q=3$ is again special but (2.27) yields

$$\frac{1}{\kappa} \sim (1-y) + 2e^{-\kappa} ,$$

so

$$\frac{1}{\kappa} \sim (1-y) + 2 \exp - (1-y)^{-1} . \quad (2.31)$$

3. The functions $b_q(\kappa)$, $B_q(\kappa)$ and its inverse.

Here we consider the functions

$$b_q(\kappa) = z_v(\kappa) = \int_{-1}^1 e^{\kappa t^2} (1-t^2)^v dt, \quad (3.1)$$

$$B_q(\kappa) = N_v(\kappa) = b'_q/b_q = z'_v/z_v. \quad (3.2)$$

where $v=(q-3)/2 \geq -\frac{1}{2}$ and $-\infty < \kappa < \infty$.

The functions $z_v(\kappa)$, $b_q(\kappa)$ and their successive derivatives form descending sequences because $|t| \leq 1$. Further

$$0 \leq b_{q_1} < b_{q_2} \quad \text{if } q_1 > q_2,$$

$$0 \leq z_{v_1} < z_{v_2} \quad \text{if } v_1 > v_2.$$

Thus

$$0 \leq N_v(\kappa) = B_q(\kappa) \leq 1 \quad (3.3)$$

Elementary calculations show that

$$z'_v = z_v - z_{v+1}, \quad (v \geq -\frac{1}{2}), \quad (3.4)$$

$$2\kappa z'_v = 2v z_{v-1} - (2v+1)z_v, \quad (v \geq 0), \quad (3.5)$$

$$2\kappa z''_v + (2v+3 - 2\kappa)z'_v - z_v = 0, \quad (v \geq -\frac{1}{2}). \quad (3.6)$$

In this case we will proceed without identifying and further describing the function $z_v(\kappa) = b_q(\kappa)$. Details are given in the Appendix.

From Cauchy's inequality and

$$B_q'(\kappa) = b_q''/b_q - (b_q'/b_q)^2 \quad (3.7)$$

$$b_q''(\kappa) < b_q(\kappa) \quad ,$$

it follows that

$$0 \leq B_q'(\kappa) \leq 1 - B_q^2(\kappa) \quad (3.8)$$

If the random vector x has the distribution (2) and $u = (\mu'x)^2$, then $1 \geq Eu \geq Eu^2 \geq \dots \geq 0$ and

$$B_q(\kappa) = Eu \quad (3.9)$$

$$B_q'(\kappa) = Eu^2 - (Eu)^2 = \text{var } u \quad (3.10)$$

$$B_q''(\kappa) = Eu^3 - Eu^2Eu - 2Eu(Eu^2 - (Eu)^2) = E(u - Eu)^3 \quad (3.11)$$

From the skewness of the distribution of u it follows that

$$B_q''(\kappa) \geq 0 \quad (\kappa \leq 0) \quad B_q''(\kappa) \leq 0 \quad (\kappa \geq 0) \quad (3.12)$$

Thus $B_q(\kappa)$ is non-decreasing and concave on $(-\infty, 0)$, convex on $(0, \infty)$, taking its minimum at $-\infty$ and its maximum at $+\infty$ in the range $[0, 1]$.

If we put $\kappa=0$ in (3.4), and in (3.5) with $v+1$ instead of v , we find that

$$B_q(0) = 1/q \quad (3.13)$$

Then (3.4) shows that

$$B_q'(0) = 2/q(q+2) . \quad (3.14)$$

From (3.4) and (3.5) we find the recurrence relation

$$B_q(\kappa)(1-B_{q-2}(\kappa)) = -\frac{1}{2\kappa} + \frac{q-2}{2\kappa} B_{q-2}(\kappa) \quad (3.15)$$

Now $B_q(\kappa)$ and $B_{q-2}(\kappa)$ tend to limits (the same from (3.4)) in $[\frac{1}{q}, 1]$ as $\kappa \rightarrow \infty$ which (3.15) shows to be

$$B_q(\infty) = 1 \quad (3.16)$$

Similarly $B_q(\kappa)$ lies in $[0, \frac{1}{q}]$ for $\kappa \leq 0$ and

$$B_q(-\infty) = 0 \quad (3.17)$$

Finally we note that $B_q(\kappa)$ satisfies the Riccati equation

$$1 - 2\kappa B_q'(\kappa) - 2\kappa B_q^2(\kappa) = (q-2\kappa)B_q(\kappa) \quad (3.18)$$

which holds for $q \geq 2$.

To find expansions for $B_q(\kappa)$ as $\kappa \rightarrow 0$, $\kappa \rightarrow +\infty$, and $\kappa \rightarrow -\infty$, we may here use either the difference equation (3.15) or the differential equation (3.18). We know of no explicit forms for special values of q but we will discuss the cases $q=2$ and $q=3$ separately. After harder calculations than the last section we find:

for $\kappa \rightarrow 0$

$$B_q(\kappa) = \frac{1}{q} + \frac{2(q-1)}{q^2(q+2)} \kappa + \frac{4(q-1)(q-2)}{q^3(q+2)(q+4)} \kappa^2 + \left\{ \frac{-8(q-1)^2}{q^4(q+2)(q+6)} \right. \\ \left. + \frac{8(q-1)(q-2)^2}{q^4(q+2)(q+4)(q+6)} \right\} \kappa^3 + O(\kappa^4) , \quad (3.19)$$

for $\kappa \rightarrow +\infty$

$$B_q(\kappa) = 1 - \frac{q-1}{2} \frac{1}{\kappa} - \frac{q-1}{4} \frac{1}{\kappa^2} - \frac{(q-1)(q+2)}{8} \frac{1}{\kappa^3} + o\left(\frac{1}{\kappa^4}\right), \quad (3.20)$$

for $\kappa \rightarrow -\infty$

$$B_q(\kappa) = -\frac{1}{2} \frac{1}{\kappa} - \frac{(q-3)}{4} \frac{1}{\kappa^2} - \frac{(q-3)(q-4)}{8} \frac{1}{\kappa^3} + o\left(\frac{1}{\kappa^4}\right) \quad (3.21)$$

Finally we need expansions for the solution of $\kappa=B_q(y)$ for y near zero, near $1/q$ and near unity. We may revert the appropriate expansions above or rewrite (3.18) as

$$\{1 - 2\kappa y^2 + (2\kappa - q)y\} \frac{d\kappa}{dy} = 2\kappa \quad (3.22)$$

For a solution when y is near q^{-1} , the needs to be changed to $z = y - q^{-1}$. We find

$$\kappa = \frac{q^2(q+2)}{2} \left(y - \frac{1}{q}\right) - \frac{(q-2)(q+2)^2}{2q(q+4)} \left(y - \frac{1}{q}\right)^2 + o\left(y - \frac{1}{q}\right)^3 \quad (3.23)$$

As $y \rightarrow 1$, $\kappa \rightarrow \infty$ and we have

$$\begin{aligned} \frac{1}{\kappa} &= \frac{2}{(q-1)}(1-y) - \frac{2}{(q-1)^2} (1-y)^2 \\ &- \frac{4(q+1)}{(q-1)^3} (1-y)^3 + o((1-y)^4) \end{aligned} \quad (3.24)$$

As $y \rightarrow 0$, $\kappa \rightarrow -\infty$ and we find

$$-\frac{1}{\kappa} = 2y + 2(q-3)y^2 - \frac{(q-3)(5q-19)}{4} y^3 + o(y^4) \quad (3.25)$$

The formulas (3.21), (3.26) are unhelpful when $q=3$.

For $q=2$, the density (2) is proportional to $\exp \kappa \cos^2 \theta$ and has normalizing constant $\int_0^{2\pi} \exp \kappa \cos^2 \theta d\theta$ when $\kappa > 0$, the modes are at $\theta=0, \pi$ while if $\kappa < 0$, they are at $\pi/2, 3\pi/2$. Thus there is no need to use $\kappa < 0$ -- a rotation of $\pi/2$ will serve. Further since $\cos^2 \theta = \frac{1}{2}(\cos 2\theta + 1)$ $\phi=2\theta$ has the density (1) with a concentration of $\kappa/2$. Thus

$$B_2(\kappa) = \frac{1}{2}(1 + A_2(\frac{\kappa}{2})) \quad (3.26)$$

It is easily verified that this relation is consistent with A_2 satisfying (2.19) and B_2 satisfying (3.18), etc.. Thus we may check (3.19) against (2.25) and (3.20) against (2.26). In so doing we find that

$$B_2(\kappa) = \frac{1}{2} + \frac{\kappa}{8} - \frac{1}{16^2} \kappa^3 + O(\kappa^5) \quad (3.27)$$

Our κ^3 term in (3.19) was so complicated that we hesitated to give it, but this device provides a check of it.

For $q=3$,

$$B_3(\kappa) = \frac{\int_{-1}^1 t^2 e^{\kappa t^2} dt}{\int_{-1}^1 e^{\kappa t^2} dt}.$$

If $\kappa \rightarrow \infty$, set $\kappa = -\lambda$, $\lambda \rightarrow \infty$. Then setting $u = (2\lambda)^{1/2} t$

$$2\lambda B(-\lambda) \sim 1 - \frac{2(2\lambda)^{1/2} e^{-\lambda}}{\sqrt{2\pi}} \left\{ 1 - \frac{2}{(2\lambda)^{1/2}} \frac{1}{\sqrt{2\pi}} e^{-\lambda} \right\}^{-1} \quad (3.28)$$

by noting the relation to the Gaussian and the fact that

$$1 - \int_{-\infty}^x e^{-z^2/2} \frac{dz}{\sqrt{2\pi}} \sim \frac{x}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \rightarrow \infty.$$

Thus (3.28) explains (3.21) i.e. $B_3(\kappa) + (2\kappa)^{-1}$ is exponentially small as $\kappa \rightarrow \infty$. Similarly as $\kappa \rightarrow \infty$, $\kappa^{-1} + 2y$ (see (3.25)) is exponentially small.

4. The computation of $A_q(\kappa)$, $B_q(\kappa)$ and associated functions.

From (2.10) it follows that

$$A_q(\kappa) = I_{q/2}(\kappa) / I_{q/2-1}(\kappa) \quad (4.1)$$

Since these modified Bessel functions are tabulated (see e.g. Abramowitz and Stegun (1964)), $A_q(\kappa)$ may be found for any q and κ . To solve $y=A_q(\kappa)$ we may use Newton's method since (2.19),

$$A'_q(\kappa) = 1 - A_q^2(\kappa) - \frac{q-1}{\kappa} A_q(\kappa) \quad (4.2)$$

and (4.1) enable us to compute the derivative. Thus hand calculations are not a problem for $A_q(\kappa)$.

To produce tables of the $A_q(\kappa)$, machine calculations are needed. If q is odd, we may first compute

$$A_3(\kappa) = \coth \kappa - \kappa^{-1} \quad (4.3)$$

and then use the recurrence relation (2.20) so that

$$A_5(\kappa) = (A_3(\kappa))^{-1} - 3\kappa^{-1}, \text{ etc..}$$

If q is odd, we need separate algorithms for $A_2(\kappa)$ and $A_4(\kappa)$ before (2.20) may be used.

To compute $A_2(\kappa)$ and $A_4(\kappa)$ one may return to the basic definition and use numerical integration to evaluate $a_q(\kappa)$, $a'_q(\kappa)$ for smaller values of κ until the asymptotic expansion (2.26) comes in to force and the ratio becomes awkward to handle. The latter will not happen if $a_q(\kappa)$ and $a'_q(\kappa)$ are divided by the first terms of their asymptotic expansions. From (2.10) and the expansion (Watson (1952)) of $I_\nu(\kappa)$, $\kappa \rightarrow \infty$, we have

$$a_q(\kappa) \sim (2/\kappa)^{q/2-1} e^\kappa \quad (4.4)$$

The derivative of (4.4) will be used for $a'_q(\kappa)$.

If $A_q(\kappa)$ is to be tabulated for a specific q it may be best to integrate the differential equation (4.2) numerically. To start the solution at $\kappa=0$, set

$$A_q(\kappa) = \kappa/q + \kappa a(\kappa) \quad (4.5)$$

so that $a(\kappa)$ satisfies

$$a' = -q^2 - \frac{\kappa}{2} - \frac{2\kappa a}{q} - \kappa a^2 \quad (4.6)$$

which is well behaved at the origin. Of course $a(0)=0$. When κ becomes large, (4.2) should be used directly.

In Watson (1981a) it is shown that the variance stabilizing transformation for κ -estimates is

$$g(\kappa) = \int^\kappa (A'(k))^{1/2} dk \quad (4.7)$$

in which the lower terminal is arbitrary. For $q=3$ we have found it convenient to start at unity. The table of $g(\kappa)$ is then found by numerical integration using (4.2) and (4.3); it is more accurate for values of $\kappa > 1$ than for $\kappa < 1$. Since $\kappa^s < 1$ are rare for $q=3$ in the applications we have met, this is very satisfactory. If one regularly dealt with quite large κ^s as in palaeomagnetism, it would be satisfactory to tabulate (4.7) by numerical integration with $A'_q(\kappa)$ replaced by the derivative of the asymptotic expansion (2.25) and starting at $\kappa = 10$ or 20 .

Turning now to the computation of $B_q(\kappa)$, it is shown in the Appendix that

$$B_q(\kappa) = \frac{M(3/2, (q+2)/2, \kappa)}{M(1/2, q/2, \kappa)} \quad (4.8)$$

where $M(a, b, z)$ is one of the Kummer functions. The tabulation most useful for our purposes is that in Rushton and Lang (1954). It gives both the numerator and denominator of (4.8) for $q=1$ (1) 5, $\kappa = 0.02$ (.02) .1(.1) 1(1) 10(10) 50, 100, 200. These tables may be extended to negative κ by using Kummer's transformation (A7).

The only known case where (4.8) reduces to more familiar functions is $q=3$, $\kappa < 0$. By specializing (4.8) or a direct calculation,

$$b_3(\kappa) = \kappa^{-1/2} \int_0^\kappa e^u u^{-1/2} du \quad (\kappa > 0) \quad (4.9)$$

and, setting for $\kappa < 0$, $-\kappa = \lambda$,

$$b_3(\kappa) = \lambda^{-1/2} \int_0^\lambda e^{-u} u^{-1/2} du \quad (\kappa < 0) \quad (4.10)$$

Thus when $\kappa < 0$, $b_3(\kappa)$ and $b_3'(\kappa)$ can be found using tables of the incomplete gamma-function.

To compute $B_q(\kappa)$ for specific $q \neq 3$ and κ , or to make tables, it seems that the best method is always to integrate the differential equation (3.18). It may be rewritten as

$$B_q'(\kappa) = -\frac{1}{2\kappa} - B_q^2(\kappa) - \left(\frac{q}{2\kappa} - 1\right) B_q(\kappa) \quad (4.11)$$

To start the solution it is essential to use (3.19) since (4.11) is badly behaved for small κ .

To solve $y = B_q(\kappa)$, when none of the expansions (3.19), (3.20) and (3.21) may be used, one must compute $B_q(\cdot)$ and $B_q'(\kappa)$ from (4.11).

To find the variance stabilizing transformation

$$h(\kappa) = \int^{\kappa} (B_q'(k))^{\frac{1}{2}} dk \quad (4.12)$$

the range of integration should avoid $k=0$ if possible.

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APPENDIX

It was shown in Section 3 that $b_q(\kappa)$ satisfied the differential equation (3.6), or

$$\kappa b_q''(\kappa) + \left(\frac{q}{2} - \kappa\right) b_q'(\kappa) - \frac{1}{2} b_q(\kappa) = 0 \quad (\text{A.1})$$

This is Kummer's differential equation -- see e.g. Chapter 13 by L.J. Slater in Abramowitz and Stegun (1964). It is self-adjoint and a confluent form of the hypergeometric equation with a regular singularity at $\kappa=0$ and an irregular singularity at $\kappa=\infty$. All regular solutions at the origin of

$$z\omega'' + (b-z)\omega' - a\omega = 0 \quad (\text{A.2})$$

are proportional to

$$M(a, b, z) = \sum_{r=0}^{\infty} \frac{(a)_r}{(b)_r} \frac{z^r}{r!} \quad (\text{A.3})$$

where

$$(a)_r = a(a+1)\dots(a+r-1)$$

Since

$$b_q(\kappa) = \int_{-1}^1 e^{\kappa t^2} (1-t^2)^{(q-3)/2} dt$$

the results of Section 1 show that

$$\begin{aligned} b_q(0) &= \omega_q / \omega_{q-1} \\ &= \pi^{1/2} \frac{\Gamma((q-1)/2)}{\Gamma(q/2)} \end{aligned} \quad (\text{A.4})$$

Thus

$${}_q b_q(\kappa) = \frac{\pi^{1/2} \Gamma((q-1)/2)}{\Gamma(q/2)} M\left(\frac{1}{2}, \frac{q}{2}, \kappa\right) \quad (\text{A.5})$$

We have already seen a reflection of some of the many recurrence relations satisfied by $M(a,b,z)$ but not the useful differential property,

$$\frac{d^n}{dz^n} M(a,b,z) = M(a+n, b+n, z) \quad (\text{A.6})$$

or the Kummer transformations

$$M(a,b,z) = e^z M(b-a, b, -z) \quad (\text{A.7})$$

$$M(1+a-b, 2-b, z) = e^z M(1-a, 2-b, -z) \quad (\text{A.8})$$

The Kummer function M is related to the Bessel function I_ν which arose in Section 2 by the formula

$$\begin{aligned} M(a,b,z) &= e^{z/2} \Gamma(b-a-\frac{1}{2}) (z/4)^{a-b+\frac{1}{2}} \\ &= \sum_{r=0}^{\infty} \frac{(2b-2a-1)_r (b-2a)_r (-1)^r}{r! (b)_r} I_{b-a+\frac{1}{2}+r}(z/2) \end{aligned} \quad (\text{A.9})$$

For $|z|$ large,

$$\begin{aligned} \frac{M(a,b,z)}{\Gamma(b)} &= \frac{e^{\pm i\pi a} z^{-a}}{\Gamma(b-a)} \left\{ \sum_{r=0}^{R-1} \frac{(a)_r (1+a-b)_r}{r!} (-z)^{-r} + O(|z|^{-R}) \right\} \\ &+ \frac{e^z z^{a-b}}{\Gamma(a)} \left\{ \sum_{s=0}^{S-1} \frac{(b-a)_s (1-a)_s}{s!} z^{-s} + O(|z|^{-S}) \right\} \end{aligned} \quad (\text{A.10})$$

where the upper sign is taken if $-\pi/2 < \arg z < 3\pi/2$, the lower otherwise.

From (A.5) and (A.6), it follows that

$$\begin{aligned}
 B_q(\kappa) &= b_q^-(\kappa) / b_q(\kappa) \\
 &= \frac{d}{d\kappa} \log M(\frac{1}{2}, q/2, \kappa) \\
 &= \frac{M(3/2, (q+2)/2, \kappa)}{M(1/2, q/2, \kappa)} \quad (A.11)
 \end{aligned}$$

From (A.11) and (A.7), we see that

$$B_q(-\kappa) = \frac{M((q-1)/2, (q+2)/2, \kappa)}{M((q-1)/2, q/2, \kappa)} \quad (A.12)$$

and (A.11) and (A.8) give yet another form for $B_q(-\kappa)$. (A.5) and (A.9) show that

$$\begin{aligned}
 b_q(\kappa) &= \pi^{\frac{1}{2}} \frac{\Gamma((q-1)/2) \Gamma((q-2)/2)}{\Gamma(q/2)} e^{\kappa/2} \left(\frac{\kappa}{4}\right)^{-(\frac{q-2}{2})} \\
 &= \sum_{r=0}^{\infty} \frac{(q-2)_r ((q-2)/2)_r (-1)^r}{r! (q/2)_r} I_{q/2+r}(\kappa/2) \quad (A.13)
 \end{aligned}$$

(A.5) and (A.13) show that, as $\kappa \rightarrow +\infty$,

$$\begin{aligned}
 b_q(\kappa) &= \pi^{\frac{1}{2}} \Gamma((q-1)/2) e^{\kappa} \kappa^{-(q-1)/2} \\
 &\left\{ \sum_{s=0}^{S-1} \frac{((q-1)/2)_s (1/2)_s}{s!} \kappa^{-s} + O(\kappa^{-S}) \right\} \quad (A.14)
 \end{aligned}$$

while if $\kappa \rightarrow -\infty$,

$$b_q(\kappa) = \pi^{\frac{1}{2}} \Gamma((q-1)/2) (-\kappa)^{-\frac{1}{2}}$$

$$\left\{ \sum_{r=0}^{R-1} \frac{(\frac{1}{2})_r (-(q-1)/2)_r}{r!} (-\kappa)^{-r} + O(\kappa^{-R}) \right\} \quad (A.15)$$

The expansions (A.13), (A.14), and (A.15) could be used directly to obtain the related expansions for $B_q(\kappa)$. However the procedure used in Section 3 is now seen to be much simpler e.g. it does not require this Appendix.

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If the unit vector x in \mathbb{R}^q has a probability density proportional to $\exp \kappa \mu^* x$, or $\exp \kappa (\mu^* x)^2$, then the statistical theory for x depends largely upon the logarithmic derivatives $A_q(\kappa)$ and $B_q(\kappa)$ of

$$\int_{-1}^1 e^{\kappa t} (1-t^2)^v dt \quad \text{and} \quad \int_{-1}^1 e^{\kappa t^2} (1-t^2)^v dt$$

where $v=(q-3)/2$. This paper gives a self-contained study of the functions $A_q(\kappa)$, $B_q(\kappa)$, of the computational problems of calculating these functions, of solving $y=A_q(\kappa)$ and $y=B_q(\kappa)$, and of finding the variance stabilizing transformations $\int^{\kappa} A^{-1}(\kappa)^{1/2} d\kappa$, $\int^{\kappa} B^{-1}(\kappa)^{1/2} d\kappa$ are also discussed.

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